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Dynamic equilibria of group vaccination strategies in a heterogeneous population

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Abstract In this paper we present an evolutionary variational inequality model of vaccination strategies games in a population with a known vaccine coverage profile over a certain time interval. The population is considered to be heterogeneous, namely its individuals are divided into a finite number of distinct population groups, where each group has different perceptions of vaccine and disease risks. Previous game theoretical analyses of vaccinating behaviour have studied the strategic interaction between individuals attempting to maximize their health states, in situations where an individual's health state depends upon the vaccination decisions of others due to the presence of herd immunity. Here we extend such analyses by applying the theory of evolutionary variational inequalities (EVI) to a (one parameter) family of generalized vaccination games. An EVI is used to provide conditions for existence of solutions (generalized Nash equilibria) for the family of vaccination games, while a projected dynamical system is used to compute approximate solutions of the EVI problem. In particular we study a population model with two groups, where the size of one group is strictly larger than the size of the other group (a majority/minority population). The smaller group is considered much less vaccination inclined than the larger group. Under these hypotheses, considering that the vaccine coverage of the entire population is measured during a vaccine scare period, we find that our model reproduces a feature of real populations: the vaccine averse minority will react immediately to a vaccine scare by dropping their strategy to a nonvaccinator one; the vaccine inclined majority does not follow a nonvaccinator strategy during the scare, although vaccination in this group decreases as well. Moreover we find that there is a delay in the majority's reaction to the scare. This is the first time EVI problems are used in the context of mathematical epidemiology. The results presented emphasize the important role played by social heterogeneity in vaccination behaviour, while also highlighting the valuable role that can be played by EVI in this area of research.

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1 Introduction

Recent studies explore the application of game theory to vaccinating behaviour under voluntary policies for childhood diseases [4,5], such as measles, mumps, chickenpox, pertussis and rubella [2]. In these papers the authors assume a homogeneous population where all individuals share the same perception of risk (of both vaccination and infection). However, in real populations, risk perception can vary significantly across distinct social groups [20,29]. In [13], the authors study the dynamics of vaccinating behaviour in a population divided into social groups, each having a different perceived risk of infection and vaccination, under a voluntary policy, via projected dynamical systems (PDS) and variational inequalities (VI), two methodologies new to mathematical epidemiology. The present paper is using some of the setup of [13], but is answering a different question.

It has been shown that whether or not an individual decides to vaccinate depends partly upon the perceived probability of their becoming infected, which in turn depends upon the level of disease prevalence [25,11,5]. Disease prevalence is a function of the vaccine coverage in the population [2], which is the collective result of the vaccination decisions of other individuals, if vaccination is voluntary. Hence, the individuals in a given population are effectively engaged in a strategic interaction (a 'game') with one another, mediated by transmission dynamics.

Vaccine scares are not uncommon and have occurred for various vaccines, including those for polio, smallpox, pertussis, measles-mumps-rubella, and Hepatitis B [1,9,20,23,34,35]. At high levels of vaccine coverage, there is a reduced individual incentive to vaccinate, since unvaccinated individuals are already protected through herd immunity. If concerns about the potential health risks of vaccination then develop, high vaccine coverage levels may be prone to destabilize, and vaccine coverage can drop rapidly. Several mathematical modelling studies have incorporated the effects of human behaviour under a voluntary policy, either explicitly or implicitly [4,5,13,21,24].

In this paper we track the evolution of a group's equilibrium vaccinating strategies in a population divided into social groups, each group having a different perceived risk of infection and vaccination. We assume an infectious disease for which vaccination can take place only shortly after birth, where parents decide on a voluntary basis to vaccinate their children, and in which individuals (children) can be either susceptible, infectious, or recovered (immune). These are known as Susceptible-Infectious-Recovered (SIR) models, and have been well-validated and widely applied in infectious disease epidemiology [2].

The modelling question we study is the following: given a finite time interval $[0, T]$, and given $p(t)$ a vaccination coverage function reflecting a vaccine scare taking place in a population over the time horizon $[0, T]$, what is the evolution of the equilibrium strategies of each population group? To answer this question we consider the concepts of generalized Nash games, quasivariational inequality (QVI) problems and evolutionary variational inequality (EVI) problems. Then using existing results from the theory of EVI we show how a solution to such games can be obtained and computed.

Historically, the first to study noncooperative behaviour was Cournot in 1838 [17]. Nash formalized and generalized these ideas in [32,33]. The fact that Nash equilibria can be reformulated as VI has been first observed in [30] in infinite dimensions. In finite dimensions we mention the works of [22,26] among many others. It is also known that a generalized Nash game can be reformulated as a QVI (noted first in [6]). In this work we use existing results to reformulate a generalized Nash vaccination game as a finite-dimensional quasivariational inequality (see [27] and the references therein.) We take one step further and introduce a (one parameter) family of generalized Nash games whose solutions can be shown to exist via an evolutionary variational inequality problem. We remark that the solutions obtained from an EVI problem may not be the only solutions of this family of games.

Evolutionary variational inequalities have been introduced in the 1960's ([30,10]), and have been used in the study of partial differential equations and boundary value problems. They are part of the general variational inequalities theory, with important applications in operations research, economics theory and transportation science (see [14,19] and the references therein). The existence and uniqueness theory for EVI problems has been answered in many contexts[3]; here we use the result in [18]). In [14,16] the authors present computational procedures for obtaining approximate solutions of an EVI problem of the type considered here. We are using one of these in our numerical example of Sect. 4. In general, an EVI provides a tool for studying equilibrium states of an applied problem whose constraints are varying with a parameter, most commonly taken to be physical time. Mathematically, a solution of an EVI problem is a curve whose individual points represent equilibrium states of the underlying problem; in brief, knowing how the constraints of an applied problem change with respect to the parameter, we are able to predict, using an EVI, how the equilibrium states of the problem will change with respect to the same parameter. The parameter considered is usually time. In this paper we take this parameter to also mean time.

The structure of the paper is as follows: Sect. 2 presents the generic context for a generalized Nash vaccination game. Section 3 introduces a time-dependent family of games, and it shows how a solution to such games is obtained with the help of an EVI problem. Finally Sect. 4 discusses the case of a family of generalized Nash games used to study our original modelling problem, namely that of tracking the time evolution of the equilibrium strategies for each population group where a vaccine scare takes place within a given time horizon $[0, T]$. A numerical example involving a minority/majority group is provided.

2 Generalized Nash vaccination games

2.1 The context of vaccination games

Strategic interactions among groups of a population under a voluntary vaccination policy have been considered and studied previously in [4] and [13]. In both of these works, the concept of a non-cooperative game was used as a model. In this subsection we present a generalized Nash game (or as is sometimes called a pseudo-Nash game) formulation of the groups' interaction. A generalized Nash game (GNE) is a noncooperative game where each player's strategy set depends on the other players' strategies. The GNE we present below is built similarly to the one in [13], but with one important difference: here we consider p, the vaccine coverage in the overall population, to be known from, for example, statistical observations.

We therefore let N denote the number of individuals in a population divided into k distinct groups and p fixed. The division is made according to the assumption that all individuals within a group share the same relative risk perception, however distinct groups have distinct relative risk perceptions. We consider a disease for which there is lifelong natural immunity, and in which individuals are typically infected early in life in the absence of vaccination (this describes the so-called paediatric infectious diseases [2]). Likewise we consider a vaccine which is administered primarily in the youngest age classes, and in which vaccination coverage is typically low later in life.

We denote by $P_i, i \in \{1, 2, \ldots, k\}$ the vaccination strategy corresponding to the i-th group (the probability that a child in group i is vaccinated) and by $\epsilon_i N$ the number of individuals in group i choosing strategy P_i. In this context

$$\epsilon_i \in (0, 1) \quad \text{and} \quad \sum_{i=1}^{k} \epsilon_i = 1.$$

We remark here that we are not interested in $\epsilon_i = 0$. If this is true for some $i \in \{1, 2, \ldots, k\}$, then the problem is reduced to a population with $k - 1$ or less distinct groups. We also note that if there exists i with $\epsilon_i = 1$, then the problem is reduced to that of a population where all individuals share the same risk assessment. This represents the social homogeneous case considered in previous work [4].

As discussed in the Introduction, the decision to vaccinate depends partly upon the perceived risks associated with infection and vaccination. The perceived probability of significant morbidity due to vaccination is denoted by r_v. The perceived probability of becoming infected given that a proportion p of the population is vaccinated, is denoted by π_p, and the perceived probability of significant morbidity upon infection is denoted r_{inf}. The overall perceived probability of experiencing significant morbidity because of not vaccinating is thus $\pi_p r_{inf}$. We denote by $r_i := r_v^i / r_{inf}^i$ the relative perceived risk of vaccination versus infection of group i. We study cases with $r_i \neq r_j,\ \forall i, j \in \{1, 2, \ldots, k\}$, otherwise the problem reduces to the case of a population with $k - 1$ or less distinct groups.

In order to find a mathematical expression for π_p, one approach is to use equilibrium solutions of a deterministic SIR compartmental model and assume that individuals have perfect knowledge of their probability of eventually becoming infected [4]. However, individuals do not have perfect knowledge of their probability of being infected. One could, for instance, assume that the perceived probability of eventually becoming infected increases linearly with the current prevalence of disease in the population [5,13]. As in these previous works, we consider that the overall vaccine coverage in the population can be expressed as $p = \sum_{i=1}^{k} \epsilon_i P_i$, where p is a fixed, known value. We assume for ease of analysis that π_p is a decreasing function of p given by $\pi_p = b/a + p$. This expresses the fact that disease prevalence is implicitly a function of how many individuals have been vaccinated, and that greater perceived coverage in the population means a reduced perceived infection risk for susceptible individuals. Because appropriate data are generally lacking on perceived risks of vaccination and infection, the validity of this function cannot be tested. However, we use values of a and b that are guided by epidemiologic constraints to ensure plausible results (see Sect. 4.2).

2.2 The generalized Nash vaccination game

In order to setup this type of game, the following facts are assumed:

1. p is considered known.
2. All individuals within a group share the same relative risk perception, however distinct groups have distinct relative risk perceptions.

The strategy set for all individuals in a group $i \in \{1, \dots, k\}$ is $\{P_i | P_i \in [0, \min\{1, p\}]\}$ and P_i is the probability that a child in group i is vaccinated. We will denote by $P = (P_1, \dots, P_k) \in \mathbb{R}^k$ the vector of strategies for all groups and by $p \in \mathbb{R}_+$ the vaccine coverage level. Let us denote for each $i \in \{1, \dots, k\}$ the vector $\hat{P}_i := (P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_k)$, $\forall P$, and let us introduce the set-valued mappings

$$K^i : [0, \min\{1, p\}]^{k-1} \to 2^{[0, \min\{1, p\}]},$$

where

$$K^i(\hat{P}) = \left\{ P_i \mid 0 \le P_i \le \min\{1, p\}, \quad P_i = \frac{p - \sum_{j=1, j \neq i}^{k} \epsilon_j P_j}{\epsilon_i} \right\}.$$

The payoff function of a group i where the perceived relative risk is r_i is given by

$$u^i : graph(K^i) \to \mathbb{R},\ u^i(P) = -r_i P_i - \pi_p(1 - P_i) = -r_i P_i - \frac{b(1 - P_i)}{a + \sum_{i=1}^{k} \epsilon_i P_i}. \quad (1)$$

The players in a given round of the game are the parents of a given cohort of children, who play the game only once (they can decide only once whether or not to vaccinate their child). Future rounds of the game are played by the parents of later cohorts. Let us define the set-valued mapping $K := \times_{i=1}^{k} K^i$, where $K(P) = \times_{i=1}^{k} K^i(\hat{P}_i)$. Following [27] we have:

Definition 2.1 The generalized Nash vaccination game is defined by the data $\{[0, \min\{1, p\}],\ K^i,\ u^i\}_{i \in \{1,\dots,k\}}$ and an equilibrium is defined by a point $P^* \in [0, \min\{1, p\}]^k$ so that

$$P_i^* \in K^i(\hat{P}_i^*),\ \forall i \in \{1, \dots, k\} \text{ and } u^i(P^*) \ge u^i(Q^i, \hat{P}_i^*),$$
$$\forall Q_i \in K^i(\hat{P}_i^*),\ \forall i \in \{1, \dots, k\}. \quad (2)$$

2.3 Generalized Nash games and quasivariational inequalities

In this section we recall the definitions of finite-dimensional variational and quasivariational inequality problems and show how a generalized Nash game as introduced above can be reformulated as a quasivariational inequality problem.

Definition 2.2 1. Let $K \subset \mathbb{R}^k$ be a closed, convex, nonempty set, $f : K \to \mathbb{R}^k$. *A variational inequality problem* given by f and K:

$$\text{finding } x \in K \text{ so that } < f(x), y - x > \ge 0, \quad \text{for all } y \in K, \quad (3)$$

where $< \cdot, \cdot >$ is the inner product on $\mathbb{R}^k$, defined by $< x, y >= \sum_{i=1}^{k} x_i y_i$, for any $x, y \in \mathbb{K}$.
2. Let $g : \mathbb{R}^k \to 2^{\mathbb{R}^k}$ a set valued mapping and $f : K \to \mathbb{R}^k$. *A quasivariational inequality problem* given by g and f is:

$$\text{finding } x \in g(x) \text{ so that } < f(x), y - x > \ge 0, \quad \text{for all } y \in g(x). \quad (4)$$

The name quasivariational inequalities was introduced in [7] and [8]. The fact that Nash equilibria can be reformulated as VI has been first observed in [30] in infinite dimensions and in [22,26] in finite dimensions. It is known that a GNE can be reformulated as a QVI [6].

We present here the QVI reformulation of the generalized vaccination game from Definition 2.1 above, inspired by [27]. Let

$$F_i(P) := -\frac{\partial u^i(P_i, \hat{P}_i)}{\partial P_i} = r_i - \pi_p - \frac{b\epsilon_i(1-P_i)}{(a+\sum_{i=1}^k \epsilon_i P_i)^2} \text{ and } F(P) = (F_1(P), \ldots, F_k(P)).$$

Since $u^i(\cdot, \hat{P}^i)$ are concave and of class $\mathcal{C}^1$, then GNE of Definition 2.1 can be reformulated as the QVI problem

$$\text{find } P^* \in K(P^*) \text{ so that } < F(P^*), \quad Q - P^* >\geq 0, \quad \forall Q \in K(P^*). \tag{5}$$

Existence of solutions to QVI problem (5) is known (see for example [27]). Uniqueness of solutions is a much more difficult matter; however, we are not concerned with this question in this paper. One way of obtaining solutions to QVI problem (5) is to define a variational inequality whose solutions are solutions to QVI. We use this approach next for the vaccination game proposed above. Let us define the set

$$\tilde{K} := \{P \mid 0 \leq P_i \leq \min\{1, p\}, \quad \sum_{i=1}^k \epsilon_i P_i = p, \quad \forall i \in \{1, \ldots, k\}\}. \tag{6}$$

We have that (see [27] for a proof):

Theorem 2.1 *Since u_i are concave and of class $\mathcal{C}^1$ in P_i, then every solution of the variational inequality*

$$\textit{find } P^* \in \tilde{K} \textit{ s. t. } < F(P^*), \ Q - P^* >\geq 0, \quad \forall Q \in \tilde{K}, \tag{7}$$

is a solution of the QVI (5) *and thus a solution of the game* (2). *The converse is not generally true.*

3 Time-dependent vaccination games

We consider as above a population with N individuals divided into k distinct groups. We assume that the vaccination coverage level in the population is measured from statistical observations over a longer time period. We shall represent this mathematically by a given function $p(t)$. For example, it has been observed that in the case of a vaccine scare (namely cases where the population becomes worried about possible side effects of a vaccine) the vaccination coverage level in the population decreases dramatically and it only recovers very slowly after the scare has ended.

In this section we want to answer the following question: given a known vaccination coverage profile (function) over a finite time interval $[0, T]$, $p : [0, T] \to \mathbb{R}_+$, what are the equilibrium vaccination strategies of various population groups corresponding to this profile?

In the last section we showed for instance that for a $t \in [0, T]$, a solution to the vaccination game (2) in a heterogeneous population can be obtained by solving a variational inequality problem of the type (7).

To answer our question posed above we introduce a 1-parameter (where the parameter is interpreted as physical time) family of generalized Nash vaccination games and we show that a solution of each of these games can be obtained by solving an evolutionary variational inequality (EVI) problem (see Subsect. 3.2 below) over $[0, T]$.

3.1 Time-dependent GNE

We consider a time-dependent family of games where each player (group) i has a vaccination strategy function $P_i \in L^2([0,T],\mathbb{R})$, $P_i : [0,T] \to [0, \min\{1, p(t)\}]$, $i \in \{1,2,\ldots,k\}$, where we assume that the vaccination coverage of the entire population is given by a piecewise continuous function p in $L^2([0,T],\mathbb{R}_+)$. As before $\epsilon_i N$ is the number of individuals in group i choosing strategy $P_i(t)$ at $t \in [0,T]$. Let us denote for each $i \in \{1,\ldots,k\}$ the vector functions

$$P : [0,T] \to [0, \min\{1, p(t)\}]^k,\ P(t) = (P_1(t),\ldots,P_k(t)) \text{ and}$$
$$\hat{P}_i(t) := (P_1(t),\ldots,P_{i-1}(t), P_{i+1}(t),\ldots,P_k(t)).$$

Let us define

$$\begin{aligned} A := & \Big\{\hat{P}_i \in L^2([0,T],\mathbb{R}^{k-1}) \mid 0 \le (\hat{P}_i)_j(t) \le \min\{1, p(t)\} \text{ a.a. } t, \\ & \forall j \in \{1,\ldots,i-1,i+1,\ldots,k\}\Big\}, \\ B = & \Big\{g \in L^2([0,T],\mathbb{R}) \mid 0 \le g(t) \le \min\{1, p(t)\},\ \text{a.a. t}\Big\}, \end{aligned}$$

and the set-valued mapping $\mathbb{K}^i : A \to 2^B$ given by

$$\mathbb{K}^i(\hat{P}_i) = \Bigg\{P_i \in L^2([0,T],\mathbb{R}) \mid 0 \le P_i(t) \le \min\{1, p(t)\},$$
$$P_i(t) = \frac{p(t) - \sum_{j=1,\, j\neq i}^{k} \epsilon_j P_j(t)}{\epsilon_i},\ \text{a.a. } t\Bigg\}.$$

Each player has a payoff $u_i : graph(\mathbb{K}^i) \to L^2([0,T],\mathbb{R})$ given by

$$u_i(P) = -rP_i - \frac{b}{a + p(t)}(1 - P_i).$$

Analogous to Sect. 2 above, let us define the set-valued mappings $\mathbb{K} := \times_{i=1}^k \mathbb{K}^i$, where $\mathbb{K}(P) = \times_{i=1}^k \mathbb{K}^i(\hat{P})$. We now introduce:

Definition 3.1 A time-dependent generalized Nash vaccination equilibrium is defined by a curve $P^* \in L^2([0,T],\mathbb{R}^k)$, so that $\forall i \in \{1,\ldots,k\}$ and a.e. on $[0,T]$:

$$\begin{aligned} & P_i^*(t) \in K^i(\hat{P}_i^*)(t),\ \text{and} \\ & u^i(P^*(t)) \ge u^i(Q^i(t), \hat{P}_i^*(t)),\quad \forall Q_i(t) \in K^i(\hat{P}_i^*)(t). \end{aligned} \tag{8}$$

We show next that an equilibrium defined by (8) can be obtained via an evolutionary variational inequality problem.

3.2 EVI

Evolutionary variational inequalities were originally introduced by Lions and Stampacchia [30] and Brezis [10] to solve problems arising from mechanics. We consider a nonempty, convex, closed, bounded subset of the reflexive Banach space $L^p([0,T],\mathbb{R}^q)$. Recall that

$$\ll \phi, x \gg := \int_0^T \langle \phi(x)(t), x(t)\rangle \mathrm{d}t$$

is the duality mapping on $L^p([0,T],\mathbb{R}^q)$, where $\phi \in (L^p([0,T],\mathbb{R}^q))^*$ and $x \in L^p([0,T],\mathbb{R}^q)$. Let $F : \mathbb{K} \to (L^p([0,T],\mathbb{R}^q))^*$; the standard form of the EVI we work with is therefore:

$$\text{find } x \in \mathbb{K} \text{ such that } \ll F(x), y-x \gg \geq 0, \ \forall y \in \mathbb{K}. \tag{9}$$

In this paper we take $p := 2$ and we denote by $\mathbb{K}$ the closed, convex, bounded subset of the Hilbert space $L^2([0,T],\mathbb{R}^k)$ given by:

$$\mathbb{K} = \Bigg\{ P \in L^2([0,T],\mathbb{R}^k) \mid 0 \leq P_i(t) \leq \min\{1, p(t)\}, \\ \forall i \in \{1,\ldots,k\}, \sum_{i=1}^{k} \epsilon_i P_i(t) = p(t) \, a.a. \, t \in [0,T] \Bigg\}. \tag{10}$$

We show next that a time-dependent Nash equilibrium (8) can be obtained from a solution of an evolutionary variational inequality.

Theorem 3.1 *Any solution $P^* \in \mathbb{K}$ of EVI* (9) *on the set* (10) *is a time-dependent generalized Nash equilibrium* (8)*, where $F(P) = \left(-\frac{\partial u_1(P)}{\partial P_1}, \cdots, -\frac{\partial u_k(P)}{\partial P_k}\right)$.*

Proof We want to show that any solution P^* of EVI (9) satisfies (8). We show equivalently that if $P^* \in L^2([0,T],\mathbb{R}^k)$ does not satisfy (8), then it is not a solution of EVI (9).

Let then $P^* \in L^2([0,T],\mathbb{R}^k)$. Since P^* is not an equilibrium (8), this implies that $\exists i \in \{1,\ldots,k\}$ and a set of positive measure E so that a.e. on E we have either

$$P_i^*(t) \notin \mathbb{K}^i(\hat{P}_i^*), \tag{i}$$

or

$$\exists\, Q_i(t) \in \mathbb{K}^i(\hat{P}_i^*)(t) \text{ with } u^i(P^*(t)) < u^i(Q_i(t), \hat{P}_i^*(t)). \tag{ii}$$

If (i) holds, then $P^* \notin \mathbb{K}$ and so it cannot be a solution of EVI (9). Alternatively, if (ii) holds then a.e. on E, $P^*(t)$ is not a solution of the QVI (5) on $\mathbb{K}^i(\hat{P}_i^*)(t)$. By Theorem 2.1, then $P^*(t)$ is not a solution of the VI (7) on the set

$$\mathbb{K}(t) = \left\{ P(t) \in \mathbb{R}^k \mid 0 \leq P_i(t) \leq \min\{1, p(t)\}, \sum_{i=1}^{k} \epsilon_i P_i(t) = p(t) \right\}.$$

However, it is known (see [31]) that the variational inequality

$$\text{find } P(t) \in \mathbb{K}(t) \text{ s.t. } < F(P(t)), Q(t) - P(t) > \geq 0, \quad \forall Q(t) \in \mathbb{K}(t), \ a.e. \ on \, [0,T]$$

is equivalent to EVI (9). From the last two statements we obtain that P^* is not a solution of EVI (9) and the proof is complete. □

Next we present results that assure existence and uniqueness of solutions to generic EVI problems, as well as a computational method for obtaining approximate solutions to the time-dependent generalized Nash game.

4 Computation of time-dependent equilibrium strategies

4.1 Existence and uniqueness of solutions for EVI

In this section, we present existence and uniqueness results for solutions to EVI problems of type (9). The following results hold (see [18] for a proof of (1), see [15] for a proof of (2), and see [12] for a proof of (3)):

Theorem 4.1 (1) *Let $p := 2$. If F in* (9) *satisfies the following conditions: F is pseudo-monotone and hemicontinuous along line segments, then EVI problem* (9) *admits a solution over the constraint set $\mathbb{K}$.*

(2) *Uniqueness of solutions is obtained if F is strictly pseudo-monotone on $\mathbb{K}$.*

(3) *Assuming the hypotheses of* (1) *and* (2) *above, if ρ is a piecewise function, then EVI* (9) *admits a unique piecewise solution.*

In our context, EVI (9) has the mapping F given by

$$F_i(P(t)) := r_i - \frac{b}{a+p(t)} - \frac{b\epsilon_i(1-P_i(t))}{(a+p(t))^2} \text{ and } F(P(t)) = (F_1(P(t)), \ldots, F_k(P(t)).$$

Lemma 4.1 The mapping F defined above is strongly monotone on $\mathbb{K}$, namely, there exists $\mu > 0$ such that

$$<< F(P) - F(Q), P - Q >> \geq \mu ||P - Q||_{L^2([0,T],\mathbb{R}^k)}, \quad \forall P, Q \in \mathbb{K}.$$

Proof From the definition of F we have that for a.a. $t \in [0, T]$:

$$< F(P(t)) - F(Q(t)), P(t) - Q(t) >= \frac{b\epsilon_i}{(a+p(t))^2} < P(t) - Q(t), P(t) - Q(t) > .$$

Denote by $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_k\}$ and by $p = \max_{t\in[0,T]} p(t)$. Then

$$\int_0^T < F(P(t)) - F(Q(t)), P(t) - Q(t) > dt \geq \frac{b\epsilon}{(a+p)^2}$$
$$\int_0^T < P(t) - Q(t), P(t) - Q(t) > \mathrm{d}t,$$

and by letting $\mu := b\epsilon/(a+p)^2$, the proof is complete. □

Now from Theorem 4.1 we deduce that EVI (9) in the case of vaccination strategies has a unique solution. In order to compute an approximation to the actual curve of group equilibria we follow the procedure outlined in [14]. In the example below we discretize the time interval $[0, T]$ and solve for the unique solution of the finite-dimensional EVI problem at t, using a projected dynamical system approach.

4.2 Example

We consider a population with 2 groups, where the majority is much more vaccination inclined than the minority. This is achieved by taking $\epsilon_1 > \epsilon_2$ and $r_1 < r_2$ (in fact: $\epsilon_1 = 0.7$, $\epsilon_2 := 0.3$, $r_1 = 0.37$ and $r_2 = 0.8$). We define a and b in the expression for π_p as follows: the parameter a determines the sensitivity of the perceived probability of infection to the vaccine coverage, i.e., large values of a imply a population where the perceived probability of infection depends

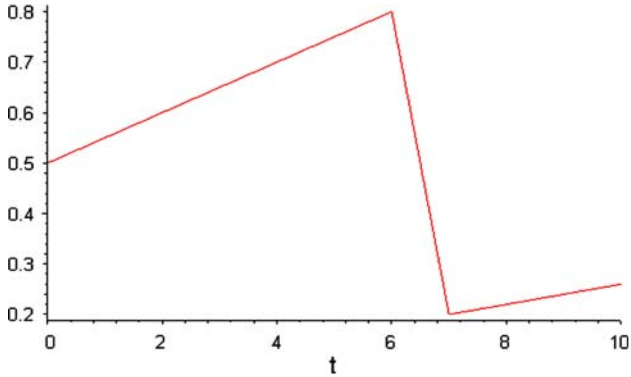

Fig. 1 Here we represented the vaccination coverage p in the overall population over the 10 yrs period considered

weakly upon the vaccine coverage, whereas small values of a imply strong dependence. The parameter combination $\frac{a}{b}$ is the maximum possible perceived probability of infection, achieved at $p = 0$. Clearly, we must have $0 < b < a$ so that $\pi_0 < 1$. For a disease such as measles, the probability that an individual eventually gets infected in the absence of any vaccination programme is close to 90% [2]. Hence, when $p = 0$, we set $\pi_0 = b/a = 0.90$. Likewise, when $p \approx p_{crit}$, where p_{crit} is the critical coverage level required to eradicate a disease, then $\pi_{p_{crit}} \approx 0$, hence we require that $a \ll p_{crit}$, and $b \ll p_{crit}$. For measles, $p_{crit} \approx 0.9$. With $a = 0.1$, and $b = 0.09$, we are consistent with these restrictions. Significantly smaller values for a and b would yield unrealistic behaviour for intermediate values of p.

In practice, one can observe the vaccine coverage level for the entire population for a given period of time. Therefore over some time interval [0, 10] where the time unit is years, we can consider that the vaccine coverage in the population has the following form (see also Fig. 1):

$$p(t) = \begin{cases} 0.5 + 0.05t & \text{if } t \in [0, 6] \\ -0.6t + 5.7 & \text{if } t \in (6, 7] \\ 0.02t + 0.12 & \text{if } t \in (7, 10]. \end{cases}$$

Note that at year 6 we modelled a vaccine scare in the whole population. Generally it is observed that vaccination coverage drops sharply when a vaccine scare happens, and it recovers much slower afterwards. In this example the constraint set is

$$\mathbb{K} = \{P \in L^2([0, T], \mathbb{R}^2) \mid 0 \leq P_i(t) \leq \min\{1, p\} \quad \text{and} \quad \sum_{i=1}^{k} \epsilon_i P_i(t) = p(t), a.e.[0, T]\}.$$

We are now able to compute the approximate time-dependent equilibrium strategies of both the majority and the minority groups, showing how they reacted to the vaccine scare. An approximate solution curve for the majority is given by

$$P_1^*(t) = \begin{cases} \frac{0.5+0.05t}{0.7}, & t \in [0, 2.6] \\ 0.9, & t \in (2.6, 6.28] \\ \frac{-0.6t+5.7}{0.7}, & t \in (6.28, 7] \\ \frac{0.02t+0.12}{0.7}, & t \in (7, 10] \end{cases},$$

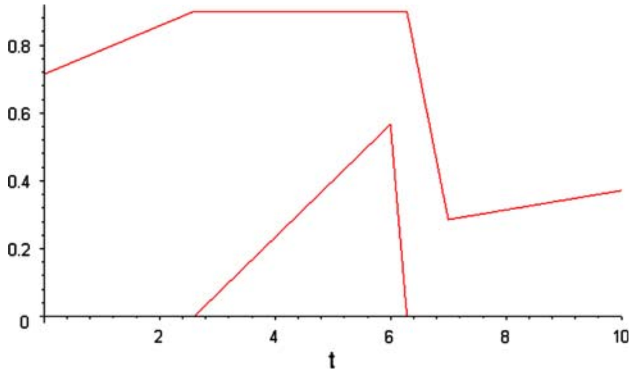

Fig. 2 The two curves represent the equilibrium strategies of the majority and minority groups over the 10 yrs period, under the vaccination coverage profile p of Fig. 1. The upper curve represents the majority's equilibrium strategies, whereas the lower curve represents the minority's equilibria

and for the minority is given by

$$P_2^*(t) = \begin{cases} 0, & t \in [0, 2.6] \\ \frac{0.05t-0.13}{0.3}, & t \in (2.6, 6] \\ \frac{-0.6t+4.07}{0.3}, & t \in (6, 6.28] \\ 0, & t \in (7, 10]. \end{cases}$$

We can see in Fig. 2 that the minority, which was less vaccination inclined to begin with, reacts immediately to the vaccine scare and their vaccination strategy decreases to 0. In turn, the majority reacts to the scare in a different manner. Given that the majority is much more vaccination inclined, its strategy does not decrease to 0 even through the vaccine scare period; we note that there is a time delay between the scare starting point $t = 6$ and the moment the majority starts reacting to the scare $t = 6.28$. This is intuitively understandable since individuals in the majority are less inclined to believe the scare right away without further information. Also, there are no time delays in the case of the minority, indicating an immediate reaction to the scare. Finally, we see that, in time, the equilibrium vaccination strategy of the majority starts to increase again, but is doing so at lower levels than during the pre-scare period.

5 Conclusions

In this paper we present a dynamic formulation of a vaccination strategies game in a heterogeneous population. We use the concepts of generalized Nash games, time-dependent generalized games and evolutionary variational inequalities to compute the game solutions (equilibrium strategies) over a finite time interval $[0, T]$, given a known vaccination coverage function in the whole population for the same interval. Vaccination games in homogeneous and heterogeneous populations have been studied before in [5] and [13], but as far as we know this is the first time that time-dependent generalized Nash games and evolutionary variational inequalities have been used in this context. We remark that one can study examples of heterogeneous populations with 3 or more groups, but we opted to include those in a future paper. We also note that different expressions for the perceived probability of infection π_p can be considered.

Game theoretical models illustrate how vaccine scares and declining vaccine coverage, especially in countries with voluntary vaccination policies such as the United Kingdom, are not isolated historical events, but rather possible instances of inherently unstable dynamics which can apply in any population under a voluntary vaccination policy. While mandatory vaccination would serve the public interest by effectively eradicating diseases, there are also implications for individual rights. Understanding and predicting long-term trends in population vaccination behaviour via game dynamic models is therefore valuable for the development of sound, evidence-based public health policy.

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